

Pion condensation in holographic QCD

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Abstract

We study pion condensation at zero temperature in a hard-wall holographic model of hadrons with isospin chemical potential. We find that the transition from the hadronic phase to the pion condensate phase is first order except in a certain limit of model parameters. Our analysis suggests that immediately across the phase boundary the condensate acts as a stiff medium approaching the Zel'dovich limit of equal energy density and pressure.

I. INTRODUCTION

Dense nuclear matter generically carries net isospin and consequently has a nonvanishing isospin chemical potential, μ_I . The Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) are presently the premier laboratories capable of studying the quark-gluon plasma phase of nuclear matter. The heavy nuclei in these experiments carry isospin, so the finite temperature system after collision has a nonvanishing isospin chemical potential. Neutron stars have still larger chemical potential, and the behavior of such objects depends on the phase structure of QCD in cold, dense environments [1–4]. At large chemical potential it is expected that mesons condense, beginning with the pions at $\mu_I \sim m_\pi$ [5–7]. The phase diagram of QCD at high density has been explored via the chiral Lagrangian [8, 9], the Nambu-Jona-Lasinio model [10–12], and lattice QCD [13–16]. The general consensus is that at low temperatures and vanishing baryon chemical potential there is a second order transition from the hadronic phase to a pion condensate phase at a critical isospin chemical potential μ_I around the pion mass $m_\pi \approx 140$ MeV, or larger at finite temperature.

The AdS/CFT correspondence [17–19] has provided motivation for extra-dimensional (holographic) models of QCD [20–28]. Both explicit and spontaneous chiral symmetry breaking may be built into the extra-dimensional models, resulting in an effective description similar to extended hidden-local-symmetry models [29, 30]. The AdS/CFT correspondence maps sources and expectation values of field theory operators to backgrounds of extra-dimensional fields with corresponding quantum numbers. By studying fluctuations about a prescribed background, the model makes predictions for field theory observables.

We study the behavior of matter with isospin chemical potential in a holographic model of hadrons. Related analyses in other holographic systems appear elsewhere (*e.g.* Refs. [31–34]). We are motivated to study the hard-wall model of Refs. [23, 24] in light of recent suggestions that the pion condensate phase is absent in that model [35]. In contrast, we identify the pion condensate phase and study its properties.

The hard-wall models [20–24] are the simplest holographic models which capture certain features of QCD. The hard-wall geometry is a region of 4+1 dimensional (5D) Anti-de Sitter space preserving the isometries of 3+1 dimensional (4D) Minkowski space. The main motivation for this choice of spacetime is its simplicity, although arguments have been made to support the choice of Anti-de Sitter space in light of asymptotic freedom at high energies

and evidence for an approximate conformal invariance in QCD at lower energies [36, 37]. The presence of a wall, which terminates the spacetime at what is referred to as the IR boundary, leads to a discrete spectrum of Kaluza-Klein modes identified via their quantum numbers with towers of hadronic resonances.

Global symmetries are lifted to gauge invariances in the holographic description. In order to include the approximate chiral symmetry of the up and down quarks, $SU(2)_L \times SU(2)_R$ gauge fields are included in the 5D model. To mimic the pattern of chiral symmetry breaking, a set of scalar fields transforming in the bifundamental representation of the chiral symmetry group is introduced. The quantum numbers of the scalar fields are those of the scalar quark bilinear $\bar{q}_L^i q_R^j$, where i and j are flavor indices which are now gauge indices. The scalar fields have a background profile which preserves the 4D Lorentz invariance of the spacetime but breaks the chiral symmetry to the diagonal isospin subgroup.

The linearized equations of motion for the scalar field have two independent background solutions: a normalizable mode and a non-normalizable mode, the difference being that the normalizable mode has a finite effective 4D action, while the non-normalizable mode has a divergent action. If the non-normalizable mode is turned on, then the theory is modified so as to explicitly break the chiral symmetry, as if by a quark mass; if the normalizable mode has a nonvanishing background then there is a spontaneous breaking of the chiral symmetry, as if by a contribution to the chiral condensate $\langle \bar{q}_L^i q_R^j \rangle$.

The chemical potential for the third component of isospin acts as a source for the isospin number density

$$N_3 = q_L^{\dagger i} T_{ij}^3 q_L^j + q_R^{\dagger i} T_{ij}^3 q_R^j, \quad (1.1)$$

which is the time component of the isospin current

$$J_V^{a\nu} = \bar{q}^i \gamma^\nu T_{ij}^a q^j, \quad (1.2)$$

where $T^a = \sigma^a/2$ are the generators of the isospin $SU(2)$, and the subscript V represents the vector subgroup of the chiral symmetry. A source for the time component of the current couples as would the time component of a background gauge field. Hence, a non-normalizable background for the time component of the vector combination of 5D gauge fields mimics an isospin chemical potential in the 4D effective theory.

As the magnitude of the isospin chemical potential is increased above the pion mass, it becomes energetically favorable for a pion condensate to form. We find that the phase

transition is first order in the hard-wall model unless the 5D gauge coupling vanishes. The speed of sound c_s at high temperatures was conjectured to satisfy a “sound bound” $c_s^2 < 1/3$ [38, 39], where $c_s^2 = 1/3$ is the conformal limit. Fluctuations in the condensate at zero temperature violate the “sound bound,” except near the phase boundary and then only if the 5D gauge coupling is small enough. Violation of the sound bound at low temperature is not unusual [39] and has also been observed in certain D-brane systems [34] and in a holographic model describing matter at a Lifshitz point [40].

To describe systems at nonvanishing temperature, extra-dimensional models are modified to include a black-hole horizon. However, we will focus on the zero-temperature phase of isospin matter, which corresponds to the original hard-wall background without a black-hole horizon. For simplicity we do not include chemical potentials except for isospin, so our analysis provides only a narrow cross section of the phase structure of the model. Extensions of these results to nonvanishing temperature and baryon chemical potential, and to include strange quarks and Kaon condensation [41], may shed light on the phases of matter in neutron stars and other extreme environments.

II. HOLOGRAPHIC PION CONDENSATION

The action for the 5D hard-wall model with chiral symmetry is given by [23, 24],

$$S = \int d^5x \sqrt{-g} \text{Tr} \left\{ |DX|^2 + 3|X|^2 - \frac{1}{4g_5^2} (F_L^2 + F_R^2) \right\}, \quad (2.1)$$

where $D_M X = \partial_M X - iL_M X + iX R_M$, $L_M = L_M^a T^a$ and $F_{MN}^L = \partial_M L_N - \partial_N L_M - i[L_M, L_N]$ (similarly for R), and we normalize the gauge kinetic term as in [23]. The spacetime in the hard-wall model is a slice of AdS_5 :

$$ds = a(z)^2 (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \quad \epsilon < z \leq z_m,$$

where $a(z) = 1/z$ in units of the AdS curvature scale, and $\eta_{\mu\nu}$ is the 4D Minkowski metric with mostly negative signature. Greek indices range from 0 to 3, and capital Latin indices from 0 to 4, with x^4 also denoted by z . The scalar fields X transform in the bifundamental representation of the $SU(2)_L \times SU(2)_R$ gauge invariance.

Chiral symmetry breaking is provided by the background solution to the X field equation of motion,

$$X_0(z) = \frac{1}{2} (m_q z + \sigma z^3) \equiv \frac{1}{2} v, \quad (2.2)$$

where m_q is the quark mass matrix responsible for sourcing σ , the chiral condensate. The bulk vector gauge field $V_M^a = 1/2(L_M^a + R_M^a)$ is dual to the isospin vector current operator. We work in the gauge $L_z^a = R_z^a = 0$. The linearized equations of motion for the transverse part of V_μ^a are

$$\partial_z \left(\frac{1}{z} \partial_z V_\mu^a \right) - \frac{1}{z} \partial_\alpha \partial^\alpha V_\mu^a = 0. \quad (2.3)$$

The background solutions for V_0^3 are of the form

$$V_0^3(z) = c_1 + \frac{c_2}{2} z^2, \quad (2.4)$$

where the coefficient of the non-normalizable mode, c_1 , is identified with the chemical potential for the third component of isospin μ_I ; and c_2 is proportional to the spontaneously generated background isospin number density, which we assume to vanish. Hence, the background gauge field is uniform,

$$V_0^3 = \mu_I. \quad (2.5)$$

The pions are identified with solutions to the linearized coupled equations of motion for the Goldstone modes in the scalar fields X , which mix with the longitudinal part of the axial vector field $A_\mu^a = (L_\mu^a - R_\mu^a)/2 \equiv \partial_\mu \phi^a$. We parametrize the Goldstone modes by fields π^a such that,

$$\begin{aligned} X &= X_0 \exp [i2\pi^a T^a] \\ &= X_0 (\cos b + i (n^a \sigma^a) \sin b), \end{aligned} \quad (2.6)$$

where $b = \sqrt{\pi^c \pi^c}$ and $n^c = b^{-1} \pi^c$. The action (2.1) takes the form:

$$\begin{aligned} S = \int d^5x \sqrt{-g} \Big\{ & 2X_0^2 (\partial_M (\cos b) \partial^M (\cos b) + \partial_M (n^a \sin b) \partial^M (n^a \sin b) \\ & - 2\mu_I a^{-2} \partial_0 (n^c \sin b) \epsilon^{a3c} n^a \sin b - 2a^{-2} \partial_\mu (n^a \sin b) \cos b \partial^\mu \phi^a \\ & + 2a^{-2} \partial_\mu (\cos b) n^a \sin b \partial^\mu \phi^a + 2\mu_I a^{-2} \cos b \epsilon^{a3c} n^a \sin b \partial_0 \phi^c \\ & + \mu_I^2 a^{-2} \sin^2 b n^c n^d \epsilon^{c3e} \epsilon^{d3e} + a^{-2} \cos^2 b \partial_\mu \phi^a \partial^\mu \phi^a + a^{-2} \sin^2 b n^a \partial_\mu \phi^a n^c \partial^\mu \phi^c) \\ & - a^{-4} \frac{1}{2g_5^2} [\mu_I^2 (\partial_i \phi^1 \partial^i \phi^1 + \partial_i \phi^2 \partial^i \phi^2) - \partial_z \partial_\mu \phi^a \partial_z \partial^\mu \phi^a + \mathcal{O}((A_\mu^a)^4)] \Big\}, \end{aligned} \quad (2.7)$$

where contractions of Greek indices are done with $\eta_{\mu\nu}$ and those of capital Latin indices are done using the full metric g_{MN} .

To quadratic order in π^a and ϕ^a , the action is

$$S = \int d^5x \left\{ v^2 a^3 \left(\frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a - \frac{1}{2} \partial_z \pi^a \partial_z \pi^a + \mu_I (\partial_0 \pi^2 \pi^1 - \partial_0 \pi^1 \pi^2) \right. \right. \\ \left. \left. + \frac{1}{2} \mu_I^2 (\pi^1 \pi^1 + \pi^2 \pi^2) - \partial_\mu \pi^a \partial^\mu \phi^a + \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \mu_I (\pi^2 \partial_0 \phi^1 - \pi^1 \partial_0 \phi^2) \right) \right. \\ \left. - a \frac{1}{2g_5^2} [\mu_I^2 (\partial_i \phi^1 \partial^i \phi^1 + \partial_i \phi^2 \partial^i \phi^2) - \partial_z \partial_\mu \phi^a \partial_z \partial^\mu \phi^a] \right\}. \quad (2.8)$$

The linearized equations of motion for π^a and ϕ^a are:

$$\begin{aligned} \phi^0 - \pi^0 &= \frac{1}{v^2 a^3 g_5^2} \partial_z (a \partial_z \phi^0), \\ m_0^2 \phi^0 - m_0^2 \pi^0 &= \frac{1}{v^2 a^3} \partial_z (v^2 a^3 \partial_z \pi^0), \end{aligned} \quad (2.9)$$

$$\begin{aligned} m_\pm \phi^\pm - (m_\pm \mp \mu_I) \pi^\pm &= \frac{m_\pm}{v^2 a^3 g_5^2} \partial_z (a \partial_z \phi^\pm), \\ (m_\pm^2 \mp \mu_I m_\pm) \phi^\pm - (m_\pm^2 \mp 2\mu_I m_\pm + \mu_I^2) \pi^\pm &= \frac{1}{v^2 a^3} \partial_z (v^2 a^3 \partial_z \pi^\pm), \end{aligned}$$

where $\pi^1 = \frac{1}{\sqrt{2}}(\pi^+ + \pi^-)$, $\pi^2 = \frac{-i}{\sqrt{2}}(\pi^+ - \pi^-)$, and $\pi^3 = \pi^0$ (similarly for ϕ^a). These equations are evaluated in the pion rest frame ($\vec{q} = 0$), identifying the pion frequency with the effective pion mass in the isospin background. Working in the gauge $A_{Lz}^a = A_{Rz}^a = 0$, the fields satisfy the boundary conditions $\partial_z \phi^{\pm,0}(z_m) = \phi^{\pm,0}(\epsilon) = \pi^{\pm,0}(\epsilon) = 0$. The boundary condition at z_m corresponds to the gauge-invariant condition $F_{z\mu}^L(z_m) = F_{z\mu}^R(z_m) = 0$, but this choice is not unique and is made for simplicity.

By eliminating $\phi^{\pm,0}$ we obtain equations of motion for $\pi^{\pm,0}$ alone.

$$\begin{aligned} \partial_z \left(\frac{1}{v^2 a^3} \partial_z (v^2 a^3 \partial_z \pi^0) \right) + m_0^2 \partial_z \pi^0 - g_5^2 v^2 a^2 \partial_z \pi^0 &= 0, \\ \partial_z \left(\frac{1}{v^2 a^3} \partial_z (v^2 a^3 \partial_z \pi^\pm) \right) + (m_\pm^2 \mp 2\mu_I m_\pm + \mu_I^2) \partial_z \pi^\pm & \\ - g_5^2 v^2 a^2 \partial_z \pi^\pm &= 0. \end{aligned} \quad (2.10)$$

Except for the replacement of the eigenvalue m_0^2 with $(m_\pm^2 \mp 2\mu_I m_\pm + \mu_I^2)$, the fields π^\pm and π^0 are solutions to the same differential equation with the same boundary conditions, with identical eigenvalues

$$m_0^2 = (m_\pm^2 \mp 2\mu_I m_\pm + \mu_I^2). \quad (2.11)$$

The neutral pion is unaffected by the chemical potential, so identifying m_0 with the pion mass in vacuum, m_π , we find a relation for the masses of the charged pions:

$$m_\pm = \pm \mu_I + m_\pi, \quad (2.12)$$

where $m_\pi \equiv m_0$. For $|\mu_I| > m_\pi$ a charged pion mass becomes negative, indicating the instability to pion condensation.

III. PROPERTIES OF THE PION CONDENSATE PHASE

Since the pattern of chiral symmetry breaking is built into our holographic model, we expect to reproduce predictions of the chiral Lagrangian, at least qualitatively. The pion effective theory is determined by the action on the solution for the pion mode discussed in the previous section, integrated over the extra dimension.

A. Decoupling the 5D gauge fields

The limit $g_5 \rightarrow 0$ provides the most direct comparison to previous results. In that limit the fluctuations of the 5D gauge fields decouple from the pion physics. The corresponding 4D effective theory is similar to the chiral Lagrangian with isospin chemical potential included as a background for a 4D isospin gauge field, as in Ref. [8]. In terms of the unitary fields

$$\Sigma = \exp \left[\frac{i\pi^a \sigma^a}{f_\pi} \right], \quad (3.1)$$

the leading order chiral Lagrangian is

$$\mathcal{L}_{4D} = \frac{f_\pi^2}{4} \text{Tr} (\nabla_\nu \Sigma \nabla^\nu \Sigma^\dagger) + \frac{m_\pi^2 f_\pi^2}{4} \text{Tr} (\Sigma + \Sigma^\dagger), \quad (3.2)$$

where $\nabla_0 \Sigma = \partial_0 \Sigma - i \frac{\mu}{2} [\sigma_3, \Sigma]$ and $\nabla_i = \partial_i$. Expanding to second order in the pion fields, the Lagrangian takes the form,

$$\begin{aligned} \mathcal{L}_{4D} = & \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a - \frac{1}{2} (m_\pi^2 - \mu_I^2) (\pi^1 \pi^1 + \pi^2 \pi^2) \\ & - \frac{1}{2} m_\pi^2 \pi^3 \pi^3 + \mu_I (\partial_t \pi^1 \pi^2 - \partial_t \pi^2 \pi^1). \end{aligned} \quad (3.3)$$

The instability when $|\mu_I| > m_\pi$ signals the phase transition to a pion condensate phase. Estimation of the value of the condensate and related observables requires an extension of the analysis to higher order in the pion fields, which we perform in the holographic description.

By design, the analysis of the 5D model is similar to the chiral Lagrangian analysis above. In the limit $g_5 \rightarrow 0$, we neglect couplings to the longitudinal gauge field $\partial_\mu \phi^a$. The action (2.7) takes the form

$$\begin{aligned} S_{g_5=0} = \int d^5x \sqrt{-g} \Big\{ & 2X_0^2 (\partial_M (\cos b) \partial^M (\cos b) + \partial_M (n^a \sin b) \partial^M (n^a \sin b)) \\ & - 2\mu_I a^{-2} \partial_0 (n^c \sin b) \epsilon^{a3c} n^a \sin b + \mu_I^2 a^{-2} \sin^2 b n^c n^d \epsilon^{c3e} \epsilon^{d3e} \Big\}, \end{aligned} \quad (3.4)$$

where $b = \sqrt{\pi^c \pi^c}$ and $n^c = b^{-1} \pi^c$ as before. The linearized equations of motion for the pion fields are now,

$$-m_\pi^2 \pi^{0,\pm} = \frac{1}{v^2 a^3} \partial_z (v^2 a^3 \partial_z \pi^{0,\pm}). \quad (3.5)$$

The condensate is a static configuration rotationally invariant in x^1, x^2, x^3 . The action on such configurations gives the condensate effective potential,

$$V_{eff, g_5=0} = \int dz v(z)^2 a(z)^3 \left(\frac{1}{2} \left(\frac{db}{dz} \right)^2 + \frac{1}{2} \sin^2 b \left(\frac{dn^c}{dz} \right)^2 - \frac{\mu_I^2}{2} \sin^2 b n^c n^d (\delta^{cd} - \delta^{c3} \delta^{d3}) \right). \quad (3.6)$$

The effective potential increases with $|dn^c/dz|$, so $dn^c/dz = 0$ in the ground state. The profile of $b(z)$ is determined from the solution to the equations of motion for the pion Kaluza-Klein mode. Expanding to fourth order in the pion fields,

$$\begin{aligned} V_{eff, g_5=0} &= \int_\epsilon^{z_m} dz v^2/z^3 \frac{1}{2} \left(\left(\frac{d\pi}{dz} \right)^2 - \mu_I^2 \left(\pi(z)^2 - \frac{\pi(z)^4}{3} + \dots \right) n^c n^d (\delta^{cd} - \delta^{c3} \delta^{d3}) \right) \\ &= \int_\epsilon^{z_m} dz v^2/z^3 \frac{1}{2} \left(m_\pi^2 \pi(z)^2 - \mu_I^2 n^c n^d (\delta^{cd} - \delta^{c3} \delta^{d3}) \left(\pi(z)^2 - \frac{\pi(z)^4}{3} + \dots \right) \right), \end{aligned} \quad (3.7)$$

where we used the linearized equation of motion (3.5) in the last line.

For $|\mu_I| > m_\pi$ it is energetically favorable to turn on the charged pions. The pion field is normalized by its kinetic term in the effective 4D theory, so we define $\pi^a(z) = \tilde{\pi}(z) \pi^a$ such that

$$\int_\epsilon^{z_m} dz v^2 a^3 \tilde{\pi}(z)^2 = 1, \quad (3.8)$$

and π^a is the pion condensate $\langle \pi^a \rangle$.

Minimizing V_{eff} expanded to $\mathcal{O}((\pi^a)^4)$, we find that the transition is smooth (second order), and for $\mu \gtrsim m_\pi$ we obtain,

$$\pi^+ \pi^- = \frac{3}{4\tilde{\eta}} \left(1 - \frac{m_\pi^2}{\mu^2} \right), \quad (3.9)$$

where $\tilde{\eta} = \int_\epsilon^{z_m} dz v^2 a^3 \tilde{\pi}(z)^4$. We then find,

$$V_{eff, g_5=0}(\pi^\pm) = -\frac{3}{8\tilde{\eta}} \mu_I^2 \left(1 - \frac{m_\pi^2}{\mu_I^2} \right)^2. \quad (3.10)$$

The isospin number density is

$$n_I = -\frac{\partial V_{eff}}{\partial \mu_I} = \frac{3\mu_I}{4\tilde{\eta}} \left(1 - \frac{m_\pi^4}{\mu_I^4} \right). \quad (3.11)$$

We can express $\tilde{\eta}$ in terms of f_π^2 in this model by the AdS/CFT determination of f_π . The correlator of a product of axial currents has a pion pole at zero momentum transfer in the chiral limit, with residue equal to f_π^2 . The AdS/CFT correspondence determines the correlation function in terms of a bulk-to-boundary propagator which solves the linearized equations of motion for the transverse part of the axial vector field. For more details in the context of the present model, see Refs. [23, 24]. We summarize the results here.

The linearized equation of motion for the transverse part of the axial vector field $A_\mu^a(q, z)$ is given by,

$$\left[\partial_z (a \partial_z A_\mu^a) + \frac{q^2}{z} A_\mu^a - v^2 a^3 g_5^2 A_\mu^a \right]_{\perp} = 0. \quad (3.12)$$

The bulk-to-boundary propagator $A(q, z)$ describes the solution to (3.12) of the form $A_\mu^a(q, z) = A(q, z) A_{0\mu}^a(q)$, with boundary conditions $\partial_z A(q, z)|_{z_m} = 0$ and $A(q, \epsilon) = 1$. In terms of the bulk-to-boundary propagator the AdS/CFT prediction for the pion decay constant is

$$f_\pi^2 = -\frac{1}{g_5^2} \left. \frac{\partial_z A(0, z)}{z} \right|_{z=\epsilon}. \quad (3.13)$$

If $g_5 = 0$ then the bulk-to-boundary propagator at $q^2 = 0$ is uniform, $A(0, z) = 1$. To next order in g_5^2 , we obtain

$$\frac{1}{z} \partial_z A(0, z) = -g_5^2 \int_z^{z_m} d\tilde{z} v(\tilde{z})^2 / \tilde{z}^3 + \mathcal{O}(g_5^4 / z_m^4). \quad (3.14)$$

From (3.13) we obtain in the $g_5 \rightarrow 0$ limit,

$$\begin{aligned} f_\pi^2 &= \int_\epsilon^{z_m} dz v(z)^2 / z^3 \\ &\approx \frac{\sigma^2 z_m^4}{4} + m_q \sigma z_m^2 + m_q^2 \log(z_m / \epsilon). \end{aligned} \quad (3.15)$$

Note that in the absence of boundary counterterms we must choose ϵ such that $\log(z_m / \epsilon) \ll \sigma z_m^2 / m_q$ to respect the chiral limit. We choose $\epsilon = 1/(10^8 \text{ MeV})$. In that case the integral defining f_π^2 is dominated by the region where the pion wavefunction $\tilde{\pi}(z)$ is approximately constant. Comparing (3.15) with (3.8), we learn that the pion wavefunction $\tilde{\pi}(z) \approx 1/f_\pi$ except for a region of small z , as in Figure 1. Similarly, from the integral defining $\tilde{\eta}$ we have

$$\tilde{\eta} \approx 1/f_\pi^2. \quad (3.16)$$

For a concrete example, fixing the mass of the lightest KK mode of the vector field V_μ^a to 776 MeV determines $z_m = 1/(323 \text{ MeV})$ [23]. Then with $m_q = 4.25 \text{ MeV}$ and chiral condensate

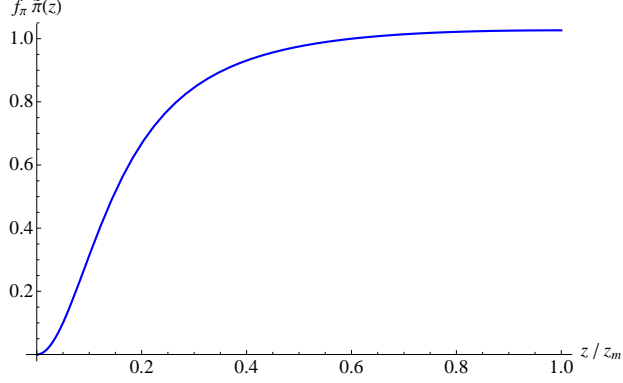


FIG. 1: Pion eigenfunction with $m_q = 4.25$ MeV, $\sigma = (263 \text{ MeV})^3$, and $z_m = 1/(323 \text{ MeV})$.

$\sigma = (263 \text{ MeV})^3$ we find physical values $m_\pi = 140 \text{ MeV}$ and $f_\pi = 92 \text{ MeV}$ in the $g_5 \rightarrow 0$ limit. With these values of the parameters, we find

$$\tilde{\eta} = \int_{\epsilon}^{z_m} dz v(z)^2 a(z)^3 \tilde{\pi}(z)^4 = 1/(91 \text{ MeV})^2, \quad (3.17)$$

which is approximately $1/f_\pi^2$ as expected. Note that the Gell-Mann-Oakes-Renner relation is approximately satisfied, $m_\pi^2 f_\pi^2 / (2m_q \sigma) = 1.07 \approx 1$.

We now have the holographic prediction of the equation of state:

$$n_I \approx \frac{3}{4} f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4} \right). \quad (3.18)$$

For comparison, the corresponding prediction from the 4D chiral Lagrangian (3.2) is [8]

$$n_{4D} = f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4} \right), \quad (3.19)$$

which differs from the holographic prediction by an overall factor of 4/3. This overall factor drops out of the ratio of pressure to energy density and the speed of sound at zero temperature. The number densities are plotted in Figure 2. The model is not expected to be valid for $\mu_I \gtrsim m_\rho \approx 5.5 m_\pi$, but we plot the model prediction here and below over the entire range of μ_I .

The pressure p and energy density ε in the pion condensate medium are determined by $n_I(\mu_I)$ [8, 13]:

$$p(\mu_I) = \int_{m_\pi}^{\mu_I} n_I d\tilde{\mu} = \frac{3f_\pi^2 (\mu_I^2 - m_\pi^2)^2}{8\mu_I^2}, \quad (3.20)$$

$$\varepsilon(\mu_I) = \int_0^{\mu_I} \mu_I d\tilde{n} = \frac{3f_\pi^2}{8\mu_I^2} (\mu_I^2 - m_\pi^2) (\mu_I^2 + 3m_\pi^2). \quad (3.21)$$

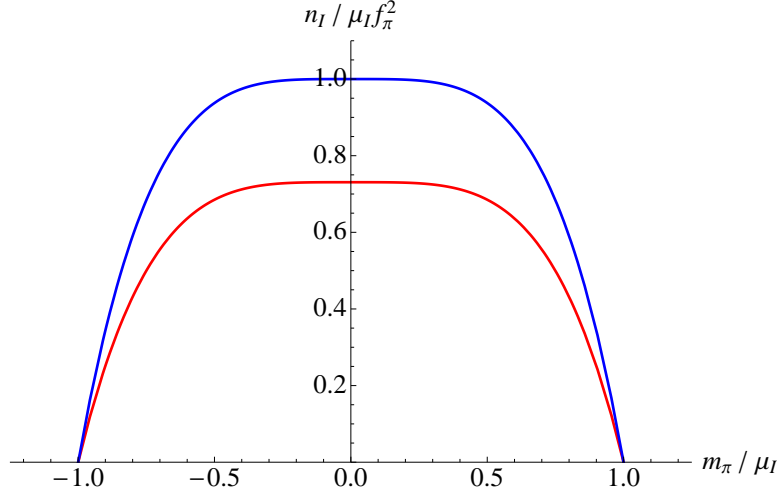


FIG. 2: Isospin number density. The bottom red curve is the prediction of the hard-wall model with $m_q = 4.25$ MeV, $\sigma = (263 \text{ MeV})^3$ and $z_m = 1/(323 \text{ MeV})$. The top blue curve is the result from Ref. [8] quoted in (3.19).

This gives

$$\frac{p}{\varepsilon} = \frac{\mu_I^2 - m_\pi^2}{\mu_I^2 + 3m_\pi^2}, \quad (3.22)$$

and

$$c_s^2 = \frac{dp}{d\varepsilon} = \frac{\mu_I^4 - m_\pi^4}{\mu_I^4 + 3m_\pi^4}. \quad (3.23)$$

The speed of sound violates the sound bound $c_s^2 < 1/3$ except near the phase transition boundary at $\mu_I = m_\pi$.

B. Pion condensation with dynamical 5D gauge bosons

Having understood how the $g_5 \rightarrow 0$ limit of holographic QCD reproduces the qualitative behavior of isospin matter at low temperature expected from the chiral Lagrangian, we now consider the more general situation including couplings to the 5D gauge fields. In the calculations below we will take $g_5 = 2\pi$, which makes the holographic prediction of the vector current polarization at large momentum transfer agree with perturbative three-color QCD [23, 24] .

We first construct an approximate solution to (2.9) as an expansion in m_π , as in Ref. [23]. Combining the first two equations of (2.9) we get

$$m_\pi^2 \partial_z \phi^0 = v^2 a^2 g_5^2 \partial_z \pi^0. \quad (3.24)$$

Recalling the boundary conditions $\pi(\epsilon) = \phi(\epsilon) = \partial_z \phi|_{z_m} = 0$, to zeroth order in m_π the solution is $\pi^0 = 0$, and ϕ^0 satisfies the same equation as the bulk-to-boundary propagator in (3.12), so we set $\phi^0(z) = A(0, z) - 1$. Away from the boundary at $z = \epsilon$ consistency of the approximate solution with the first of the equations in (2.9) requires $\pi^0 \approx -1$. Recalling that the charged pions have the same wavefunction as the neutral pion, we temporarily normalize the fields so that $\pi^\pm(z) = \pi^0(z)$. Then $\phi^\pm = (1 \mp \mu_I/m_\pm) [A(0, z) - 1]$.

Integrating the action (2.8) by parts we get,

$$\begin{aligned}
S = \int d^4x dz \left\{ \left[v^2 a^3 (\pi^+ \pi^- - \pi^+ \phi^- - \pi^- \phi^+ + \phi^+ \phi^-) - \frac{1}{g_5^2} \partial_z (a \partial_z \phi^-) \phi^+ \right] \partial_\mu \pi^+(x) \partial^\mu \pi^-(x) \right. \\
\left. + [\partial_z (v^2 a^3 \partial_z \pi^+) \pi^- + \mu_I^2 v^2 a^3 \pi^+ \pi^-] \pi^+(x) \pi^-(x) \right. \\
\left. - i [v^2 a^3 \mu_I (\pi^- \pi^+ - \phi^- \pi^+)] \partial_t \pi^-(x) \pi^+(x) + i [v^2 a^3 \mu_I (\pi^+ \pi^- - \phi^+ \pi^-)] \partial_t \pi^+(x) \pi^-(x) \right\}, \quad (3.25)
\end{aligned}$$

where functions without an argument are understood to be functions of z . Ignoring the π^0 terms, which can be obtained by taking $\mu_I \rightarrow 0$, we can use the third of Eqs. (2.9) to solve for ϕ^\pm and obtain

$$\begin{aligned}
S = \int d^4x \left\{ \left[-\frac{\mu_I^2}{m_- m_+} \int dz v^2 a^3 \pi \pi - \frac{1}{g_5^2} \int dz \partial_z (a \partial_z \phi^-) \pi \right. \right. \\
\left. \left. + \frac{\mu_I}{m_-} \frac{1}{g_5^2} \int dz \partial_z (a \partial_z \phi^+) \pi \right] \partial_\mu \pi^+(x) \partial^\mu \pi^-(x) \right. \\
\left. + \left[\frac{m_+}{g_5^2} \int dz \partial_z (a \partial_z \phi^+) \pi + \mu_I^2 \int dz v^2 a^3 \pi \pi \right] \pi^+(x) \pi^-(x) \right. \\
\left. + i \left[\frac{\mu_I}{m_-} \int dz v^2 a^3 \pi \pi + \frac{1}{g_5^2} \int dz \partial_z (a \partial_z \phi^-) \pi \right] \partial_t \pi^-(x) \pi^+(x) \right. \\
\left. - i \left[-\frac{\mu_I}{m_+} \int dz v^2 a^3 \pi \pi + \frac{1}{g_5^2} \int dz \partial_z (a \partial_z \phi^+) \pi \right] \partial_t \pi^+(x) \pi^-(x) \right\}, \quad (3.26)
\end{aligned}$$

where we have used $\pi \equiv \pi^0(z) = \pi^\pm(z)$. We now use the approximate solutions for ϕ^\pm and make the approximation $\pi = -1$ in those integrals dominated by the region where the pion wavefunction is flat. Defining $\alpha \equiv \frac{1}{f_\pi^2} \int dz v^2 a^3 \pi \pi$ and making use of equation (3.13), we get

$$\begin{aligned}
S = \int d^4x \left\{ \left(\frac{m_\pi^2 - \alpha \mu_I^2}{m_\pi^2 - \mu_I^2} \right) f_\pi^2 \partial_\mu \pi^+(x) \partial^\mu \pi^-(x) - (m_\pi^2 - \alpha \mu_I^2) f_\pi^2 \pi^+(x) \pi^-(x) \right. \\
\left. - 2i \mu_I f_\pi^2 \left(\frac{m_\pi^2 - \alpha \mu_I^2}{m_\pi^2 - \mu_I^2} \right) \partial_t \pi^+(x) \pi^-(x) \right\}. \quad (3.27)
\end{aligned}$$

The resulting Lagrangian after canonically normalizing the kinetic term is, transforming

back to (π^1, π^2, π^3) ,

$$\begin{aligned} \mathcal{L}_{eff} = & \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a - \frac{1}{2} (m_\pi^2 - \mu_I^2) (\pi^1 \pi^1 + \pi^2 \pi^2) \\ & - \frac{1}{2} m_\pi^2 \pi^3 \pi^3 + \mu_I (\partial_t \pi^1 \pi^2 - \partial_t \pi^2 \pi^1). \end{aligned} \quad (3.28)$$

This agrees with the leading order 4D chiral Lagrangian (3.3).

The effective potential for static configurations of $\pi(x)$ takes the same form as (3.6) because the longitudinal gauge bosons are derivatively coupled. We make the ansatz that $n^3 = 0$; the pion expectation value is only in the (π^1, π^2) plane. With the approximate solutions for $\pi(z)$ and $\phi(z)$ described above, and applying the canonical rescaling of the kinetic term, the effective potential at $\mathcal{O}(b^4)$ becomes

$$V_{eff}(b) = \frac{1}{2} (m_\pi^2 - \mu_I^2) b^2 + \frac{1}{6} \mu_I^2 f_\pi^{-2} \left(\frac{m_\pi^2 - \mu_I^2}{m_\pi^2 - \alpha \mu_I^2} \right)^2 \eta b^4, \quad (3.29)$$

where $\eta = \frac{1}{f_\pi^2} \int dz v^2 a^3 \pi \pi \pi \pi$, $\alpha = \frac{1}{f_\pi^2} \int dz v^2 a^3 \pi \pi$, and f_π is given by (3.13). Note that $\pi(z)$ here is the approximate solution for small m_π described earlier, and is not the canonically normalized field. We can now solve for the minimum of the potential, $\partial V_{eff} / \partial b = 0$:

$$b_0^2 = \frac{3}{2} \frac{f_\pi^2}{\eta \mu_I^2} \frac{(m_\pi^2 - \alpha \mu_I^2)^2}{(\mu_I^2 - m_\pi^2)}. \quad (3.30)$$

The isospin number density is

$$n_I = -\frac{\partial V_{eff}}{\partial \mu_I} = \mu_I f_\pi^2 \frac{1}{\eta} \frac{3}{4} \left(\alpha^2 - \frac{m_\pi^4}{\mu_I^4} \right). \quad (3.31)$$

The phase transition is first order, with the order parameter jumping at the phase boundary $\mu_I = m_\pi$. The minimum value of the free energy V_{eff} at $\mathcal{O}(b^4)$ is discontinuous at the critical point $\mu_I = m_\pi$, but the perturbative expansion breaks down near the critical point and we expect a nonperturbative analysis to confirm continuity in the free energy across the phase boundary. However, we can say with confidence that the transition is not second order in this model, because if it were then the order parameter b would vary smoothly and we would expect a perturbative analysis to be valid. Although perturbation theory breaks down near the transition, we plot the perturbative prediction for the number density in Figure 3.

The ratio of the pressure to energy density is now

$$\frac{p}{\varepsilon} = \frac{\alpha^2 \mu_I^2 - m_\pi^2}{\alpha^2 \mu_I^2 + 3m_\pi^2}, \quad (3.32)$$

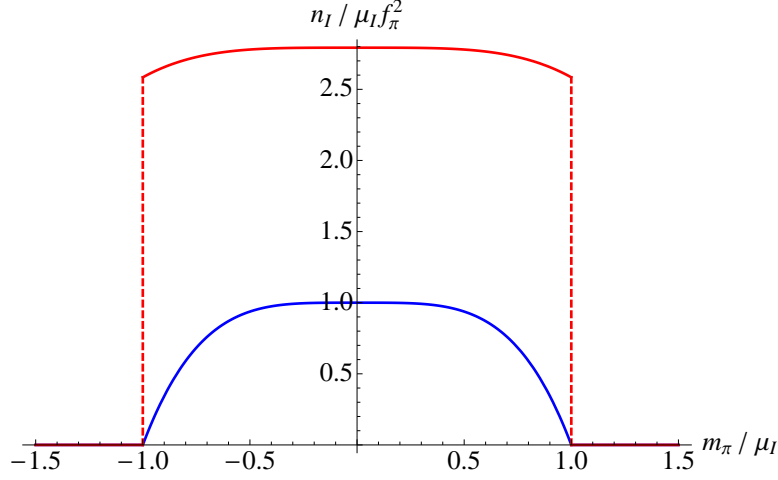


FIG. 3: Isospin number density. The top red curve is the perturbative prediction of the hard-wall model with parameters fit to m_ρ , m_π and f_π : $m_q = 2.26$ MeV, $\sigma = (333 \text{ MeV})^3$, and $z_m = 1/(323 \text{ MeV})$. This set of parameters gives $\alpha = 3.66$ and $\eta = 3.60$, in (3.31). The bottom blue curve is the result from Ref. [8] given in (3.19).

and

$$c_s^2 = \frac{\alpha^2 \mu_I^4 - m_\pi^4}{\alpha^2 \mu_I^4 + 3m_\pi^4}. \quad (3.33)$$

This is plotted next to the chiral Lagrangian prediction in Figure 4. Note that the speed of sound exceeds the sound bound $c_s^2 = 1/3$ throughout the pion condensate phase at zero temperature.

IV. A COMMENT ON THE CHIRAL LAGRANGIAN

Leading order chiral perturbation theory predicts that the transition to the pion condensate phase is second order. We have learned that gauging the chiral symmetry in the holographic model qualitatively modifies predictions for pion condensation at zero temperature. The transition becomes first order, and the medium becomes stiff immediately beyond the phase boundary. Including higher derivative terms in the chiral Lagrangian can have similar consequences, as we will now demonstrate. Consider the Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{f_\pi^2}{4} \text{Tr} [D_\mu \Sigma D^\mu \Sigma^\dagger] + \frac{m_\pi^2 f_\pi^2}{4} \text{Tr} [\Sigma + \Sigma^\dagger] + \alpha_1 (\text{Tr} [D_\mu \Sigma D^\mu \Sigma^\dagger])^2 \\ & + \alpha_2 \text{Tr} [D_\mu \Sigma D_\nu \Sigma^\dagger] \text{Tr} [D^\mu \Sigma D^\nu \Sigma^\dagger], \end{aligned} \quad (4.1)$$

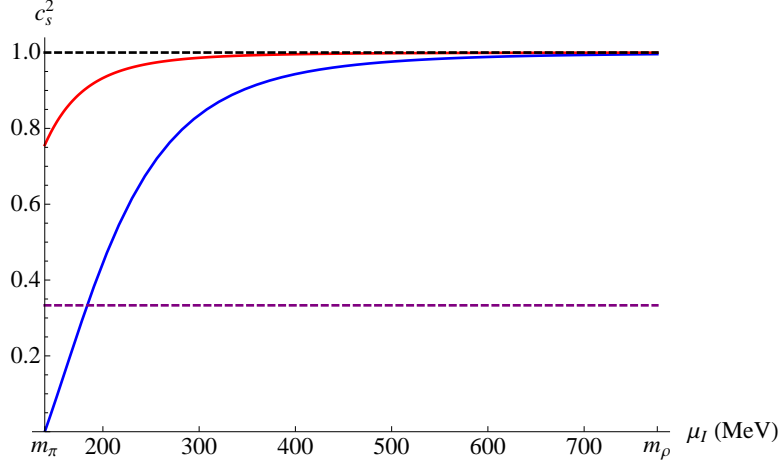


FIG. 4: Speed of Sound. The upper red curve is the perturbative prediction of the hard-wall model for the speed of sound with $m_q = 2.26$ MeV, $\sigma = (333 \text{ MeV})^3$, and $z_m = 1/(323 \text{ MeV})$. This set of parameters gives the value $\alpha = 3.66$ in (3.33). The bottom blue curve is the prediction based on (3.19). The top and bottom dashed lines represent the speed of light and the conformal limit $c_s^2 = 1/3$.

where for this analysis α_1 and α_2 are arbitrary parameters. Once again we take the static part of the Lagrangian to get an expression for the effective potential. Defining $\Sigma = \cos b + i(n^a \sigma^a) \sin b$, we have

$$V_{eff}(\cos b) = -\frac{\mu_I^2 f_\pi^2}{2} (1 - \cos^2 b) (1 - n^3 n^3) - m_\pi^2 f_\pi^2 \cos b - a_1 \frac{\mu_I^4 f_\pi^2}{4} (1 - \cos^2 b)^2 (1 - n^3 n^3)^2, \quad (4.2)$$

where $a_1 \equiv \frac{16}{f_\pi^2}(\alpha_1 + \alpha_2)$. At the minimum of V_{eff} , $n^3 = 0$, and we find a region of a_1 parameter space where the phase transition is first order. That is, as a_1 increases past a critical value $a_1^{crit} = 1/(2m_\pi^2)$, the phase transition changes from second to first order. This is illustrated in Figure 5. However, $f_\pi^2 a_1^{crit} = 0.22$ is much larger than the typical low energy coefficients in the chiral Lagrangian inferred by experiment ($l_1(m_\pi) = (-4 \pm 6) \times 10^{-3}$, $l_2(m_\pi) = (9.1 \pm 0.2) \times 10^{-3}$) [42, 43].

V. CONCLUSIONS

We have studied pion condensation at zero temperature and finite isospin chemical potential in a hard-wall model of holographic QCD with chiral symmetry breaking and massive pions. At the critical point $\mu_I = m_\pi$ the pion condenses, and our perturbative analysis

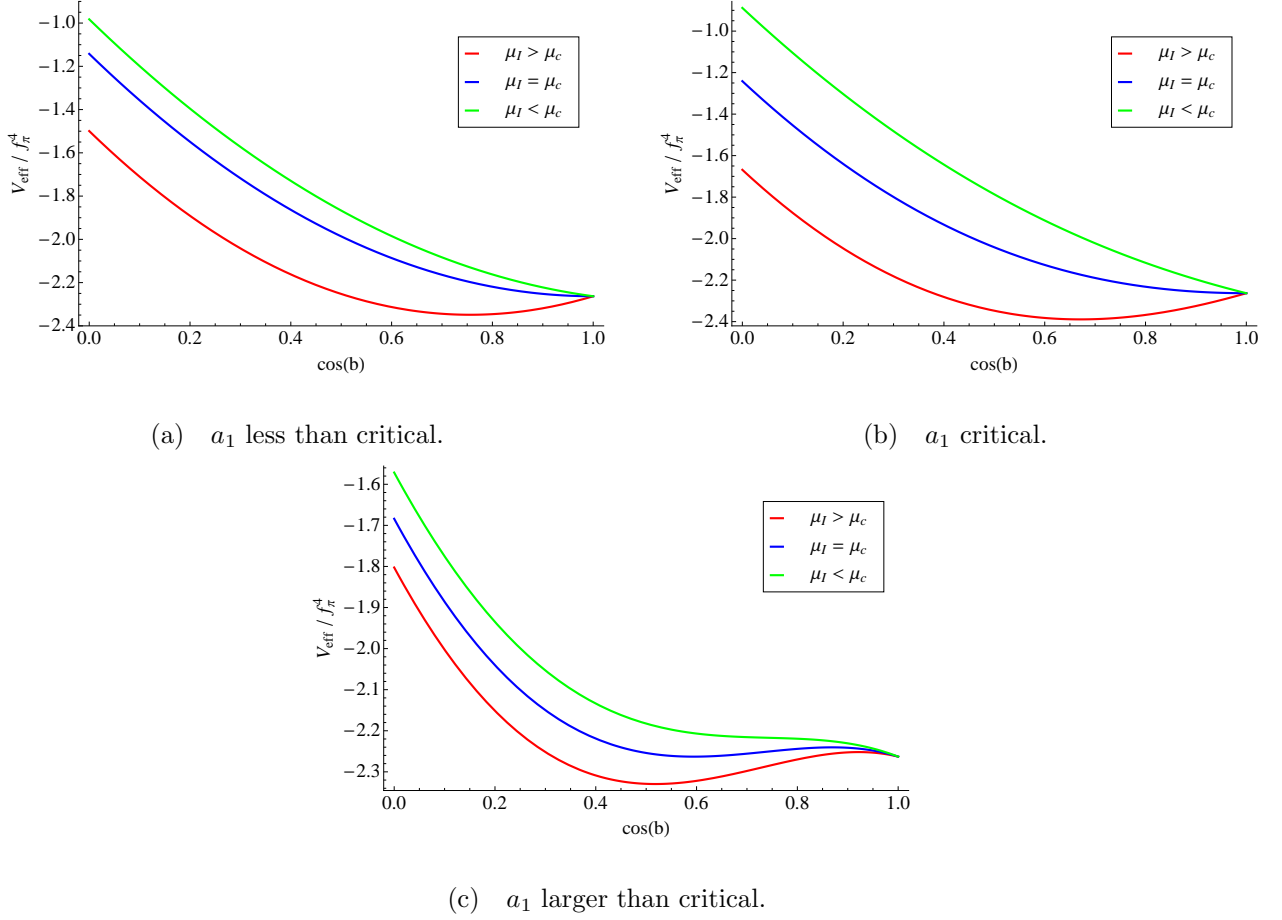


FIG. 5: Each plot shows the phase transition for a different value of the a_1 parameter. The critical value of μ_I for pion condensation depends on a_1 . The three curves shown in each plot correspond to $\mu_I < \mu_c$, $\mu_I = \mu_c$, and $\mu_I > \mu_c$ (top, middle, and bottom curves, respectively). Plot (a) shows the transition for a_1 less than the critical value. Plot (b) is with a_1 the critical value, while a_1 of (c) is larger. These plots assumed $m_\pi = 139$ MeV and $f_\pi = 92.4$ MeV.

suggests that the condensate creates a stiff medium approaching the Zel'dovich equation of state $p = \varepsilon$. Sound propagation exceeds the conformal sound bound $c_s^2 = 1/3$, except near the phase transition boundary if the 5D gauge coupling is small enough. The low-energy effective theory for pions as derived from the hard-wall model indicates that the transition from the hadronic phase to the condensate phase is first order, except in the limit of vanishing 5D gauge coupling. This is in contrast to leading order chiral perturbation theory, which predicts a second order transition [8], and lattice simulations which also seem to be consistent with a second order transition [14]. We have shown that even in chiral perturbation theory the transition can become first order if higher derivative terms in the chiral

Lagrangian have large enough coefficients.

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